Analytic Geometry

Review of geometry

Euclid laid the foundations of geometry that have been taught in schools for centuries.

In the last century, mathematicians such as Bernhard Riemann (1809 - 1900) and Nicolai Lobachevsky (1793 - 1856) transformed geometry with ideas such as curved space and spaces with higher dimensions.

Euclid’s theorems are a set of axioms that apply to flat surfaces:

- parallel lines don’t meet
- internal angles of a triangle sum to 180

As soon as the surface or space becomes curved, such rules break down.

2D analytic geometry

Equation of a straight line

The well-known equation of a line is

$$y = mx + c$$

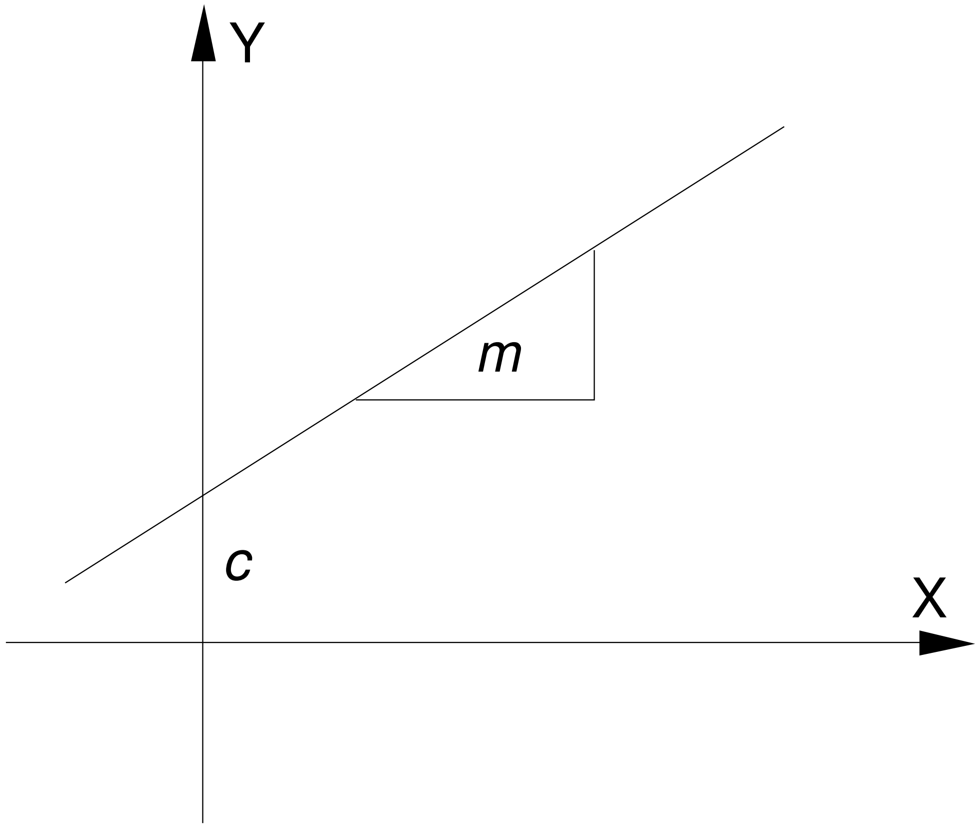

Fig. 10.18 *The normal form of the straight line is $y = mx + c$*

Line equation: normal form

Given two points (x_1, y_1) and (x_2, y_2)

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

which yields

$$y=(x-x_1)\frac{y_2-y_1}{x_2-x_1}+y_1$$

Although these equations have their uses, the more general form is much more useful:

$$ax+by+c=0$$

which possesses some interesting qualities

The Hessian normal form

The Hessian normal form is a line whose orientation is controlled by a normal unit vector $n = [a \;\; b]^T$.

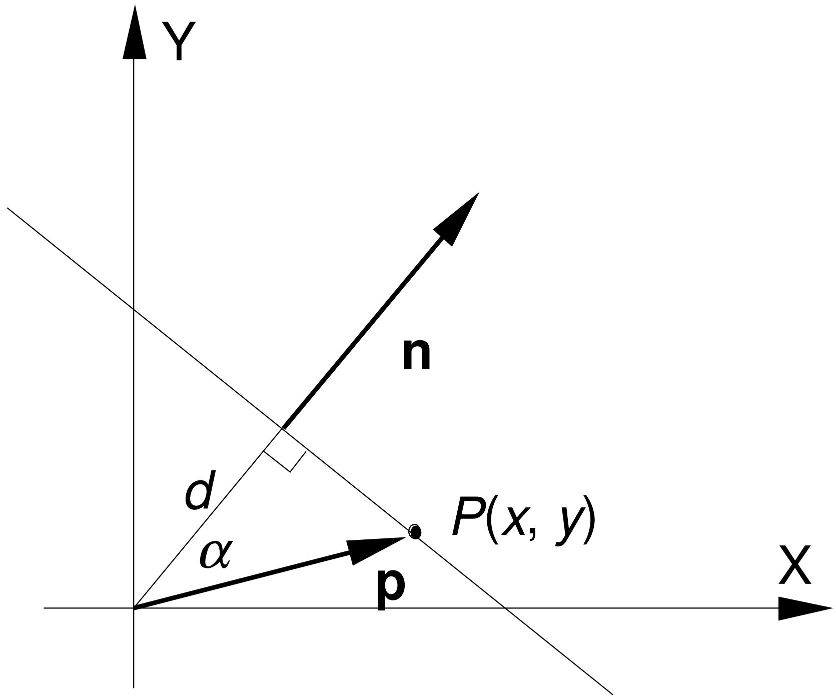

Fig. 10.19 *The orientation of a line can be controlled by a normal vector **n** and distance d.*

If $P(x, y)$ is any point on the line, then p is a position vector where $p = [x \;\; y]^T$ and d is the perpendicular distance from the origin to the line.

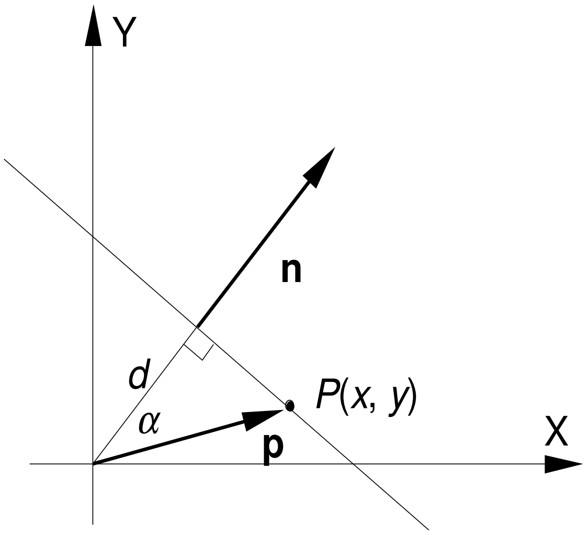

Fig. 10.19 *The orientation of a line can be controlled by a normal vector* ***n*** *and distance d.*

$$\frac{d}{|p|}=\cos(\alpha) \qquad\qquad d=|p|\cos(\alpha)$$

But the dot product $n \cdot p$ is given by

$$n \cdot p=|n||p|\cos(\alpha)=ax+by$$

therfore

$$ax+by=d|n|$$

and because $|n|=1$ we can write

$$ax+by-d=0$$

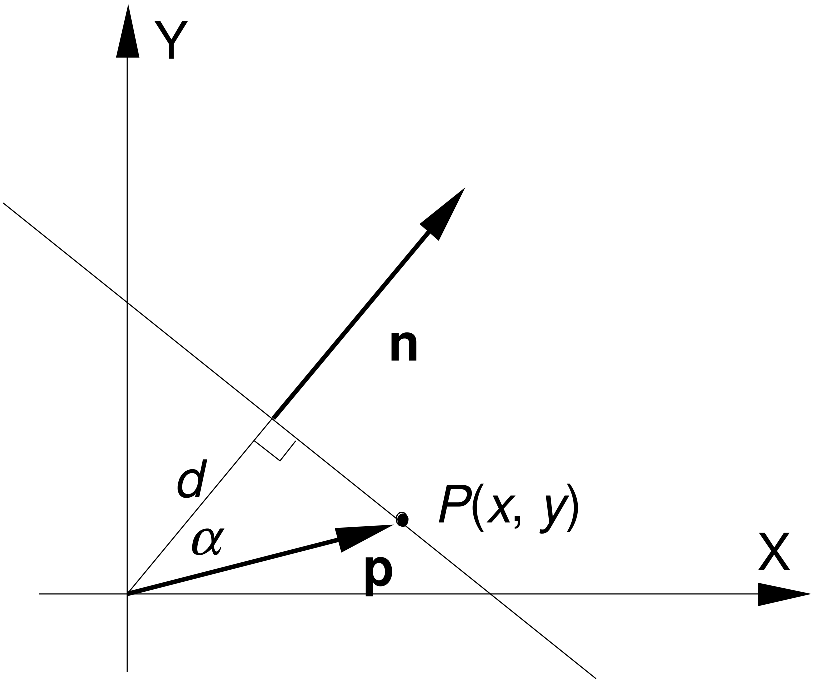

Fig. 10.19 *The orientation of a line can be controlled by a normal vector* ***n*** *and distance d.*

$$ax + by - d = 0$$

(x, y) is a point on the line
a and b are the components of a unit vector normal to the

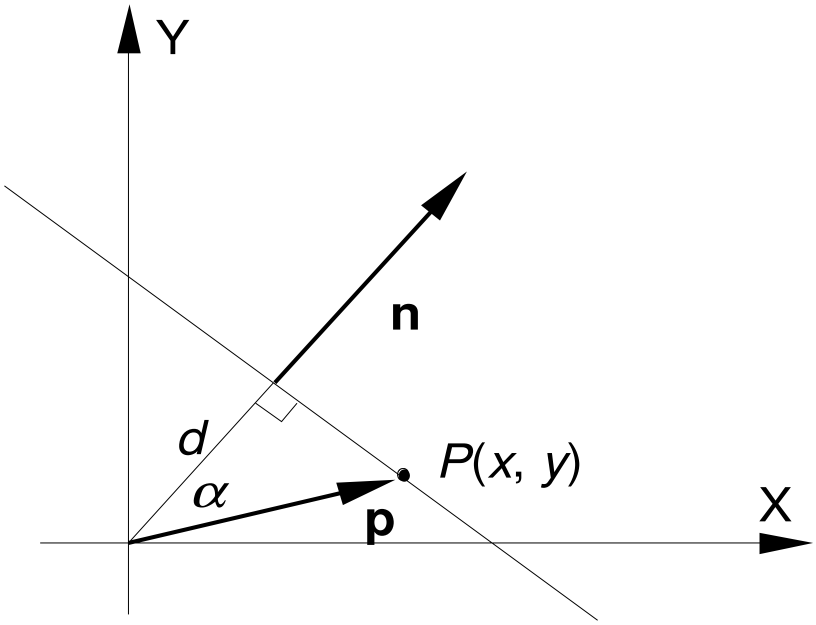

Fig. 10.19 *The orientation of a line can be controlled by a normal vector* ***n*** *and distance d.*

line
d is the perpendicular distance from the origin to the line.

d is positive when the normal vector points away from the origin, otherwise it is negative.

Example 1

Find the equation of a line whose normal vector is $[3 \;\; 4]^T$ and the perpendicular distance from the origin to the line is 1.To begin, we normalize the normal vector to its unit form.

Therefore if $n = [3 \;\; 4]^T$, $|n| = \sqrt{3^2 + 4^2} = 5$

The equation of the line is

$$\frac{3}{5}x + \frac{4}{5}y - 1 = 0$$

Example 2

Given $y = 2x + 1$, what is the Hessian normal form?

Rearranging the equation we get

$$2x - y + 1 = 0$$

If we want the normal vector to point away from the origin we multiply by -1

$$-2x + y - 1 = 0$$

Normalize the normal vector to a unit form

(i.e. $\sqrt{(-2)^2 + 1^2} = \sqrt{5}$

$$-\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y - \frac{1}{\sqrt{5}} = 0$$

Therefore, the perpendicular distance from the origin to the line, and the unit normal vector are respectively

$$\frac{1}{\sqrt{5}} \qquad \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}^T$$

The two signs from the square root provide the alternate directions of the vector, and sign of *d*.

Space partitioning

The Hessian normal form partitions space into two zones: points above the line in the partition that includes the normal vector, and points in the opposite partition.

$$ax + by - d = 0$$

If (x, y) is on the line the equation is satisfied

If (x_1, y_1) is in the partition in the direction of the normal vector, it creates the inequality

$$ax_1 + by_1 - d > 0$$

If (x_2, y_2) is in the partition opposite to the direction of the normal vector, it creates the inequality

$$ax_2 + by_2 - d < 0$$

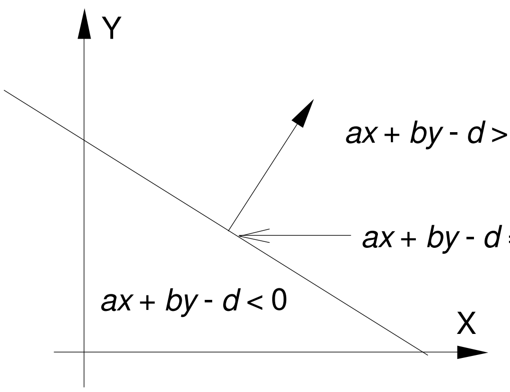

Fig. 10.20 *The Hessian normal form of the line equation partitions space into two zones*

The Hessian normal form from two points

Given (x_1, y_1) and (x_2, y_2) compute the values of a, b and d for the Hessian normal form.

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}=\frac{\Delta y}{\Delta x}$$

$$(y-y_1)\Delta x=(x-x_1)\Delta y$$

$$x\Delta y-y\Delta x-x_1\Delta y+y_1\Delta x=0$$

which is the general equation of a straight line.

For the Hessian normal form: $\sqrt{\Delta x^2+\Delta y^2}=1$.

Therefore, the Hessian normal form is given by

$$\frac{x\Delta y-y\Delta x-(x_1\Delta y-y_1\Delta x)}{\sqrt{\Delta x^2+\Delta y^2}}=0$$

Hessian Normal Form from two Points

$$\frac{x\Delta y - y\Delta x - (x_1\Delta y - y_1\Delta x)}{\sqrt{\Delta x^2 + \Delta y^2}} = 0$$

Given the following points:
$(x_1, y_1) = (0, 1)$ and $(x_2, y_2) = (1, 0)$; $\Delta x = 1$, $\Delta y = -1$

$$x(-1) - y(1) - (0 \times -1 - 1 \times 1) = 0$$

$$-x - y + 1 = 0$$

We now convert it to the Hessian normal form:

$$\frac{-x - y + 1}{\sqrt{1^2 + (-1)^2}} = \frac{-x - y + 1}{\sqrt{2}} = 0$$

$$-\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0$$

or

$$\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$

Intersection point of two straight lines

Given two line equations of the form

$$a_1 x + b_1 y + c_1 = 0$$

$$a_2 x + b_2 y + c_2 = 0$$

the intersection point (x_i, y_i) is given by

$$x_i = \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1} \quad \text{and} \quad y_i = \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}$$

If the denominator is zero, the equations are linearly dependent, indicating that there is no intersection.

Point inside a triangle

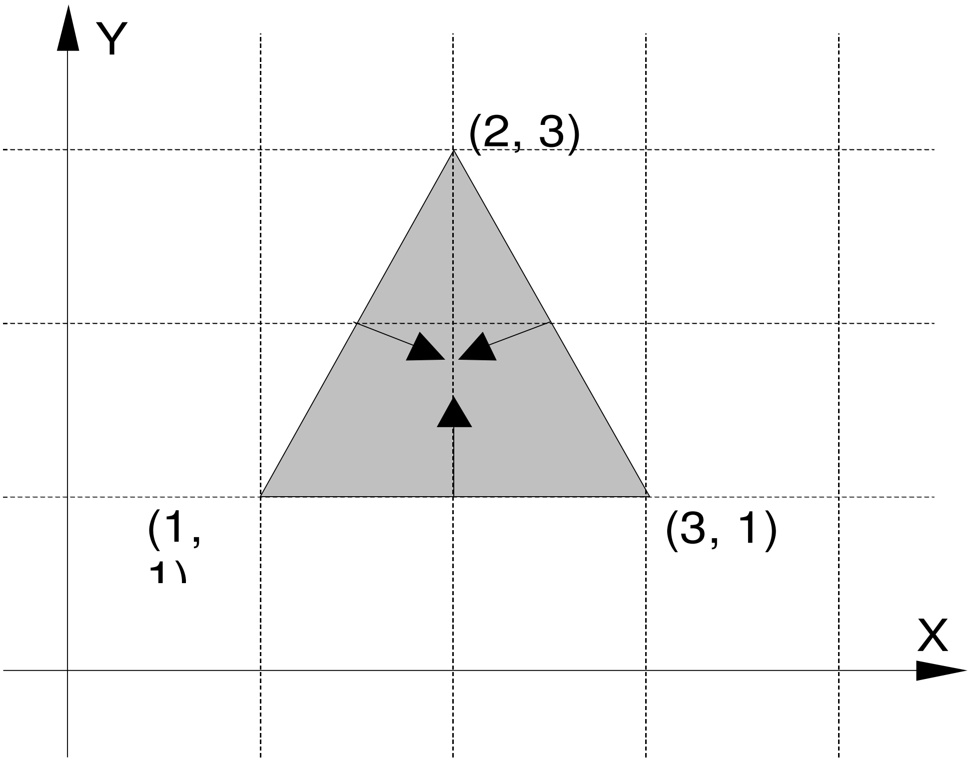

Fig. 10.26 *The triangle is formed from three line equations expressed in the Hessian normal form. Any point inside the triangle can be found by evaluating the equations*

Thus the three line equations for the triangle are

$$2y - 2 = 0$$
$$-2x - y + 7 = 0$$
$$2x - y - 1 = 0$$

We are only interested in the sign of the left-hand expressions:

$$2y - 2$$
$$-2x - y + 7$$
$$2x - y - 1$$

which can be tested for any arbitrary point (x, y).

1. **If they are all positive, the point is inside the triangle.**
2. **If one expression is negative, the point is outside.**
3. **If one expression is zero, the point is on an edge,**
4. **if two expressions are zero, the point is on a vertex.**

The point (2, 2). The three expressions are positive, which confirms that the point is inside the triangle.

The point (3, 3) is outside the triangle, which is confirmed by two positive results and one negative.

The point (2, 3), which is a vertex, creates one positive result and two zero results.

1: The line between $(1, 1)-(3, 1)$

$$0(x-1)+2(1-y)=0$$

$$-2\mathrm{y}+2=0$$

Reverse the normal vector:

$$2\mathrm{y}-2=0$$

2: The line between $(3, 1)-(2, 3)$

$$2(x-3)+(-1)(1-y)=0$$

$$2\mathrm{x}-6-1+y=0$$

$$2\mathrm{x}+y-7=0$$

Reverse the normal vector:

$$-2\mathrm{x}-y+7=0$$

3: The line between $(2, 3)-(1, 1)$

$$(-2)(x-2)+(-1)(3-y)=0$$

$$-2\mathrm{x}+4-3+y=0$$

$$-2\mathrm{x}+y+1=0$$

Reverse the normal vector:

$$2\mathrm{x}-y-1=0$$

Thus the three lines equations for the triangle are

$$2y-2=0$$

$$-2x-y+7=0$$

$$2x-y-1=0$$

We are only interested in the sign of the left-hand expressions:

Thus

$$2y-2$$

$$-2x-y+7$$

$$2x-y-1$$

which can be tested for any arbitrary point (*x,y*). If they are all positive, the point is inside the triangle. If one expression is negative, the point is outside the triangle. If one expression is zero, the point is on edge, and if two expressions are zero, the point is on a verex.

3D Geometry

Equation of a straight line

we start by using a vector **b** to define the orientation of the line.

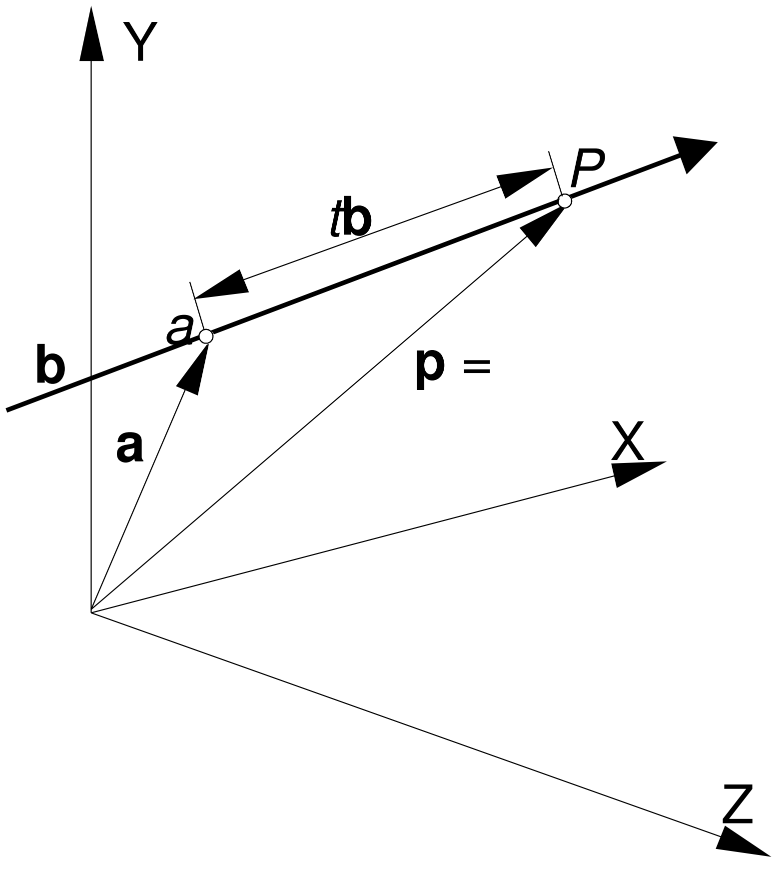

Fig. 10.28 *The line equation is based upon the point a and the vector b*

$$p = a + tb$$

Example

$$\boldsymbol{p}=\boldsymbol{a}+t\,\boldsymbol{b}$$

from which we can obtain the coordinates of the point *P*:

$$x_p=x_a+tx_b$$

$$y_p=y_a+ty_b$$

$$z_p=z_a+tz_b$$

For example, if $\boldsymbol{b}=[1 \;\; 2 \;\; 3]^T$ and $\boldsymbol{a}=(2,\,3,\,4)$ then by setting $t = 1$ we can identify a second point on the line:

$$x_p=2+1=3$$

$$y_p=3+2=5$$

$$z_p=4+3=7$$

In fact, by using different values of t we can slide up and down the line with ease.

If we already have two points in space P_1 and P_2, such as the vertices of an edge, we can represent the line equation using the above vector technique:

$$\boldsymbol{p} = \boldsymbol{p_1} + t(\boldsymbol{p_2} - \boldsymbol{p_1})$$

where $\boldsymbol{p_1}$ and $\boldsymbol{p_2}$ are position vectors to their respective points.

Once more, we can write the coordinates of any point P as follows:

$$x_p = x_1 + t(x_2 - x_1)$$

$$y_p = y_1 + t(y_2 - y_1)$$

$$z_p = z_1 + t(z_2 - z_1)$$

Equation of a plane

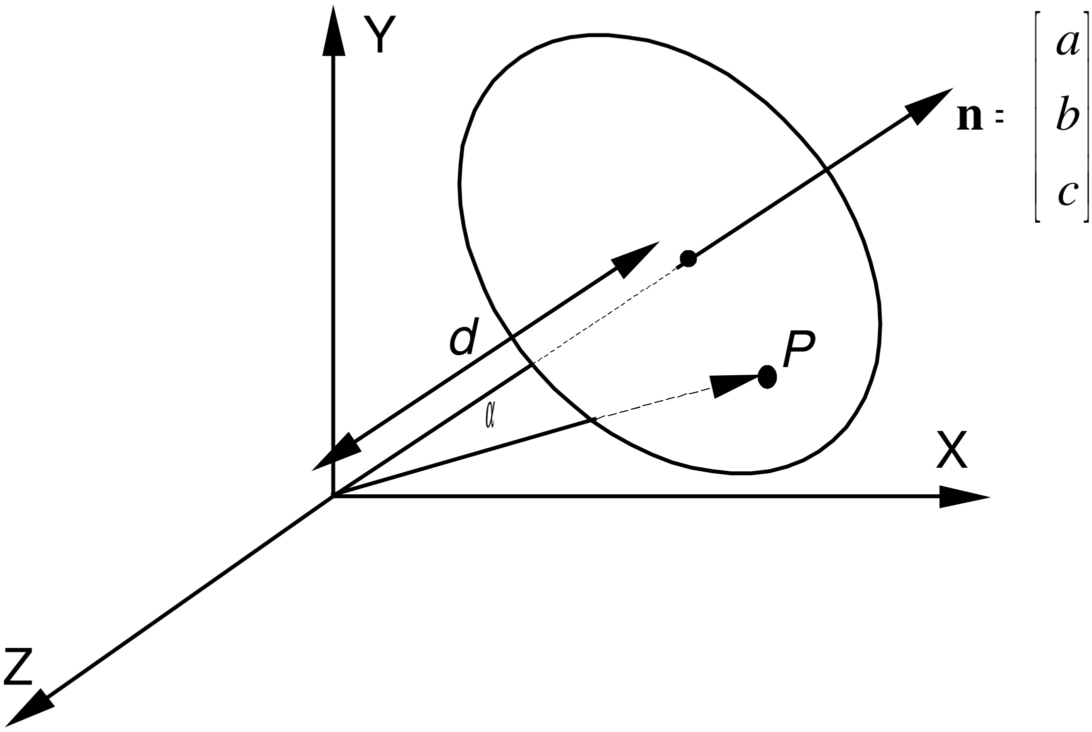

Fig. 10.29 *The Hessian normal form for a plane employs the dot product of the normal vector and the position vector to any point to create the plane equation*

$$ax + by + cz - d = 0$$

We can write

$$\frac{d}{|\boldsymbol{p}|} = \cos(\alpha)$$

and $$d = |\boldsymbol{p}|\cos(\alpha)$$

But the dot product $\boldsymbol{n} \cdot \boldsymbol{p}$ is given by

$$\boldsymbol{n} \cdot \boldsymbol{p} = |\boldsymbol{n}||\boldsymbol{p}|\cos(\alpha) = ax + by + cz$$

which implies that

$$ax + by + cz = d|\boldsymbol{n}|$$

and because $|n| = 1$ we can write

$$ax + by + cz - d = 0$$

Example

Find the equation of a plane whose normal vector is $[1\ 2\ 2]^T$ and the perpendicular distance from the origin to the plane is 1.

To begin, we normalize the normal vector to its unit form. Therefore, if $\boldsymbol{n}=\begin{bmatrix}1 & 2 & 2\end{bmatrix}^T$, $|\boldsymbol{n}|=\sqrt{1^2+2^2+2^2}=3$ and the plane equation is

$$\frac{1}{3}x+\frac{2}{3}y+\frac{2}{3}z-1=0$$

Space partitioning

When we compute the expression $ax+by+cz-d$

- points on the plane create a zero value
- points in the space partition containing the normal vector create a positive value
- points in the opposite partition yield a negative value.

For example,

$$\text{for } \frac{1}{3}x+\frac{2}{3}y+\frac{2}{3}z-1=0$$

the point (30, 30, 30) is clearly in the partition away from the origin and yields a value of +49.

the point $(0,\frac{1}{2},\frac{1}{2})$ is in the partition containing the origin yields a value of $-\frac{1}{3}$.

Thus we can easily test to see which side a point is relative to a plane.